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One loop renormalizability of supersymmetric Yang-Mills theories on noncommutative two-torus

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Abstract: We argue that Yang-Mills theory on noncommutative torus, expressed in the Fourier modes, is described by a gauge theory in a usual commutative space, the gauge group being a generalization of the area preserving diffeomorphisms to the noncommutative case. In this way, performing the loop calculations in this gauge theory in the continuum limit, we show that this theory is *one loop renormalizable*, and discuss the UV and IR limits. The moduli space of the vacua of the noncommutative super Yang-Mills theories in 2+1 dimensions is discussed.

Keywords: D-branes, M(atrix) Theories, String Duality.

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Contents

1. Introduction

It has been pointed out by A. Connes, M. Douglas and A. Schwarz (CDS) [1] that the supersymmetric Yang-Mills theory (SYM) on noncommutative torus (NCT), is naturally related to compactification of Matrix theory with a constant C_3 field. Since the eleven dimensional three-form flux is related to the NSNS two-form flux in string theory, CDS's conjecture means that the dynamics of D-branes in a B-field background is described by gauge theory on a noncommutative torus, where the B-field gives the deformation parameter of the torus, in the zero volume limit [2, 3, 4, 5, 6, 7, 8].

The noncommutative torus (NCT) is a basically flat and compact space in which the coordinates do not commute:

$$[x^i, x^j] = \frac{\theta^{ij}}{2\pi i}\,. \tag{1.1}$$

In this paper we mostly consider the two-torus and hence in our case the NCT is described by only one parameter θ which is called deformation parameter of the torus [9].

It was first explicitly shown in [4], completed and generalized in [7, 10], that the above commutation relation naturally appears for the components of open strings attached to D-brane in the B-field background. Further study on these open string dynamics and quantizing them revealed that their low energy dynamics is governed by the noncommutative supersymmetric Yang-Mills (NCSYM) theory defined on the NCT [2, 11].

Since NCSYM describes the low energy interactions of open strings at which it is decoupled from string theory dynamics and some special limits of M-theory, it is believed that NCSYM should be a well-defined quantum theory [1, 11, 12]. But, as pointed out in many papers e.g. [1, 2, 13, 14], NCSYM is a *non-local* field theory, where the non-locality scale is identified with the deformation parameter of the torus which is a *dimensionless* parameter and hence this non-locality is expected to remain at all energies; and the question of renormalizability, renormalization flow and existence of fixed points of the non-local field theories has not been answered yet. Existence of a UV fixed point, in which the theory is scale invariant, is one of the elements that make a usual field theory, e.g. a gauge theory, work. Using this fixed point, we are able to define the theory in the continuum limit, but in the NCSYM, since it is a non-local field theory, the concept of renormalization and fixed points should be understood and studied in some other sense.

In this paper, first we briefly review the algebraic structure of NCT and the gauge bundles defined on it, and discuss the Morita equivalence between sections of these bundles on different tori. This equivalence enables us to map a U(N) NCSYM with magnetic flux M to a U(1) NCSYM. So, having studied the U(1) NCSYM, we can discuss the U(N) case on the same footing.

In section 3, by using the Fourier transformation on the NCT, we map the U(1) NCSYM to a conventional commutative gauge theory, with a gauge group which is a generalization of the area-preserving diffeomorphisms of the two-torus, SDiff(T^2), to the noncommutative case. We extensively study the propagators and vertices of this gauge theory.

In section 4, performing the explicit loop calculations we show that, the related divergences up to one loop are like a usual gauge theory, in which the quadratic Casimir of the group, $C_2(G)$, is 2.

In section 5, we argue that NCYM in UV and IR limits behaves as in a usual gauge theories. In other words, although NCYM is a non-local theory, the theory is one loop renormalizable and it admits a UV fixed point. We also briefly discuss the moduli space of vacua and R-symmetries of NCSYM, as a theory with 16 real super-charges.

Finally in section 6, we will summarize our results and discuss some open and unanswered issues on this problem.

2. Noncommutative Yang-Mills theory, a review

In order to build the YM theory on a noncommutative torus, we briefly review the necessary ingredients from noncommutative geometry. For an extensive reference, we refer the reader to Connes' book [9].

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Gauge bundles on noncommutative torus. According to noncommutative geometry formulation, geometric spaces are described by a C^*-algebra which is not in general commutative. This is the generalization of the Gelfand-Nimark theorem which substitutes a compact manifold by the algebra of functions on it. In the case of noncommutative two-torus, the related C^*-algebra can be generated by the Fourier modes: $U_i = e^{2\pi i x^i}, i = 1, 2$, where

$$U_1 U_2 = e^{2\pi i\theta} U_2 U_1 \,. \tag{2.1}$$

or equivalently $[x^1, x^2] = \theta/2\pi i$. The same algebra can also be generated by the star product among these functions:

$$(f * g)(x) = \exp\left(\pi i\theta\epsilon^{ij} \frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(x)g(y)|_{y=x} \,. \tag{2.2}$$

In the above definitions θ is the deformation parameter of the NCT. We also need a set of derivatives ∂_i, which satisfy

$$[\partial_i, \partial_j] = 0 \,, \qquad [\partial_i, x^j] = \delta_i^j \,. \tag{2.3}$$

By generalizing the Serr-Swan theorem to the noncommutative spaces, gauge bundles are the finitly projective right (left) -modules over the C^*-algebra. So one can realize the connection, ∇_i, over a bundle by:

$$\nabla_i(cA) = (\nabla_i c)A + c\partial_i A \,, \tag{2.4}$$

where $c \in C^*$, $A \in$ projective right-module over C^*. According to (2.4) the gauge connection commutes with all of the elements in C^* and hence it is not a function of noncommutative x^i, but there is a combination of x^i, ∂_j [8],

$$\tilde{x}^i = x^i + \frac{i\theta}{2\pi}\epsilon^{ij}\partial_j \,, \tag{2.5}$$

which do commute with x^i and the gauge bundles and the related gauge transformations should be a function of $\tilde{x}^i$:

$$\nabla_i = \partial_i + iA_i(\tilde{x}) \,, \tag{2.6}$$

and the gauge fields $\tilde{x}$, can be expanded by Fourier modes $\tilde{U}_i = e^{2\pi i\tilde{x}^i}$;

$$\tilde{U_1}\tilde{U_2} = e^{-2\pi i\theta}\tilde{U_2}\tilde{U_1} \,. \tag{2.7}$$

Although up to here, we have discussed only the U(1) bundles, the U(N) generalization is trivially possible, by simply letting A take values in the corresponding Lie-algebra.

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It is interesting to construct the parallel string theoretic description of the mathematics we have built here. According to Witten [15], the zero modes of the open strings attached to a D-brane form a gauge multiplet of the U(1) theory living on the brane, and these gauge fields are functions of the zero modes the corresponding open strings. In the presence of a non-zero B-field background, in spite of the noncommutative coordinates, this is still valid. To see this let us consider the mode expansion of such open strings [7, 10, 11]

$$X^i = x^i + p^i\tau - B^i_j p^j \sigma + \text{Oscil.} \qquad i,j = 1,2\,, \tag{2.8}$$

where p^i are the related momentums and x^i are an arbitrary point on the tours. Quantizing these open strings, we find:

$$[p_i, p_j] = 0\,, \qquad [p_i, x^j] = -i\delta_i^j\,, \qquad [x^i, x^j] = i\frac{B^{ij}}{1+B^2}\,. \tag{2.9}$$

In the above algebra p_i can be represented by $-i\partial_i$. The coordinates of these open strings at $\sigma = \pi$ which form the corresponding gauge multiplet, is $\tilde{x}^i$ (2.5).

Following 't Hooft, we can construct the non-Abelian U(N) gauge bundles with a non-zero magnetic flux M.[1] Consider U(N) matrices:

$$V_1 V_2 = e^{2\pi i \frac{M}{N}} V_2 V_1\,. \tag{2.10}$$

The generalization of $\tilde{U}_i$, the Fourier expansion basis, to this case can be obtained by Z_i:

$$Z_1 = e^{\frac{2\pi i}{N}\tilde{x}^1} V_2^K\,, \qquad Z_2 = e^{\frac{2\pi i}{N+M\theta}\tilde{x}^2} V_1^{-K}\,, \tag{2.11}$$

where K is an integer satisfying $NL - MK = 1$, $L \in \mathbb{Z}$. one can easily check that

$$Z_1 Z_2 = e^{-2\pi i\hat{\theta}} Z_2 Z_1\,, \qquad \hat{\theta} = \frac{K + L\theta}{N + M\theta}\,. \tag{2.12}$$

Along the lines of [8], we can build the gauge connection through Z_i:

$$\exp\nabla_1^0 = e^{\partial_1} V_1 Z_2^{-M}\,, \qquad \exp\nabla_2^0 = e^{\partial_2} e^{-2\pi i \frac{M}{N}\tilde{x}^1} V_2 Z_1^M\,, \tag{2.13}$$

and

$$\nabla_i = \nabla_i^0 + iA_i(Z)\,. \tag{2.14}$$

The most important feature of the above gauge bundle construction is the Morita equivalence, which is a mathematical equivalence between the bundles on the different noncommutative tori. Comparing (2.11) and (2.1), we see that both $\tilde{U}_i$ and Z_i generate the same type of algebra; the corresponding deformation parameters are related by an Sl$(2,\mathbb{Z})$ transformation, (2.12). Hence, we can identify the *non-Abelian* gauge field A, with an *Abelian* one, $\hat{A}$, which lives on another NCT defined by $\hat{\theta}$.

[1]We assume M, N that are relatively prime.

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This equivalence between the gauge bundles on different noncommutative tori, is a special form of *Morita equivalence.*

In writing the explicit form of the NCSYM, we prefer to use the algebra of functions defined by star product, (2.2). According to CDS [1], the noncommutative U(1) gauge connection can be built by

$$\nabla_i = \partial_i + i\{A_i, \cdot\}_{M.B.} \tag{2.15}$$

with

$$\begin{aligned}\{f,g\}_{M.B.}(x) &= (f*g)(x) - (g*f)(x) \\ &= 2if(x)\sin\left(i\pi\theta\epsilon^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right)g(y)\Big|_{y=x}.\end{aligned} \tag{2.16}$$

The curvature is

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_{[\mu}A_{\nu]} + i\{A_\mu, A_\nu\}_{M.B.}\,, \qquad \mu = 0,1,2\,. \tag{2.17}$$

Then U(1) NCYM (on $R\times T^2_\theta$) is given by

$$S = \frac{1}{g^2_{YM}}\int d^3x\; F_{\mu\nu}F^{\mu\nu}\,. \tag{2.18}$$

The above action enjoys the gauge invariance:

$$A_\mu \to A_\mu + \partial_\mu\epsilon + i\{\epsilon, A_\mu\}_{M.B.}\,. \tag{2.19}$$

We can also supersymmetrize this action by adding the correct fermionic and scalar degrees of freedom [5, 16]. The maximally supersymmetric action which has 16 real super-charges is of the form:

$$\begin{aligned}S = \frac{1}{g^2_{YM}}\int d^3x\; &F_{\mu\nu}F^{\mu\nu} - 2g^2_{YM}(\nabla_\mu X^a)(\nabla^\mu X^a) + 2g^4_{YM}(\{X^a, X^b\}_{M.B.})^2 - \\ &- 2i\Theta^\alpha\Gamma^\mu_{\alpha\beta}\nabla_\mu\Theta^\beta + \frac{1}{4}g_{YM}\Theta^\alpha\Gamma^a_{\alpha\beta}\{X_a,\Theta^\beta\}_{M.B.}\,,\end{aligned} \tag{2.20}$$

where $\mu,\nu = 0,1,2,\ a,b = 3,\ldots,9$ and

$$\begin{aligned}\nabla_\mu X^a &= \partial_\mu X^a + \{A_\mu, X^a\}_{M.B.}\,, \\ \nabla_i\Theta^\alpha &= \partial_i\Theta^\alpha + \{A_i, \Theta^\alpha\}_{M.B.}\,.\end{aligned}$$

The above action is a $D=10,\ N=1$ non-Abelian Yang-Mills theory, dimensionally reduced to 2+1, with the group commutators substituted for the *Moyal bracket.*

It is worth noting that, to our knowledge, there are few theories which admit the maximally supersymmetric extension, i.e. 16 real super-charges. Among these theories, those with at most two derivatives are all dimensional reductions of ten dimensional SYM [17]. The other example of these theories are Born-Infeld actions. Despite lack of Lorentz invariance and higher derivative terms, NCSYM theories have a supersymmetric extension, and this property supports the renormalizability of these theories. Even these theories, like the SYM theories, satisfy the *non-renormalization* theorem, i.e. they are finite.

For the rational θ case, as we discussed earlier, the NCSYM, by means of Morita equivalence, is mapped to a conventional U(N) SYM theory with a magnetic flux, which we know is renormalizable. For the general real θ, we will show that by mapping the theory to the momentum space, we can analyze and study it in the conventional perturbative gauge theory language.

3. NCSYM in the momentum space

Although NCSYM has been argued to be a non-local field theory [2, 12, 13] with the non-locality scale remaining at all scales, we want to build another formulation of the theory in which the theory looks like a usual gauge theory. Remembering (2.3) or string theoretic version of it (2.9), one observes that expressing the NCYM in the ∂_i or momentum basis, or equivalently in the Fourier modes, formally removes the higher derivative terms already present in the Moyal bracket.

To see this explicitly, let us consider the Fourier expansion of the gauge field $A(\tilde{x})$ [5],

$$A_\mu(\tilde{x}) = \sum_k A_\mu(\vec{k}) L_{\vec{k}} \,, \tag{3.1}$$

where $\vec{k}$ is a vector on the *dual commutative torus*, defined in the usual manner, i.e. assume that θ is set to zero and dualize the torus, and $L_{\vec{k}}$ are the expansion basis,

$$L_{\vec{k}} = e^{ik_i\tilde{x}^i} \,. \tag{3.2}$$

In order to analyze the NCYM theory first we study the algebra defined by $L_{\vec{k}}$

$$[L_{\vec{m}}, L_{\vec{n}}] = \{L_{\vec{m}}, L_{\vec{n}}\}_{M.B.} = 2i \sin{(\pi\theta \vec{m}\times\vec{n})} L_{\vec{m}+\vec{n}} \,, \tag{3.3}$$

with $\vec{m}\times\vec{n} = \epsilon^{ij} m_i n_j$.

For the rational θ case, e.g. $\theta = 1/N$, by choosing the proper normalization for $L_{\vec{m}}$, the above algebra is

$$[L_{\vec{m}}, L_{\vec{n}}] = 2i\frac{N}{\pi} \sin\left(\frac{\pi}{N}\vec{m}\times\vec{n}\right) L_{\vec{m}+\vec{n}} \,. \tag{3.4}$$

In the *large* N limit (3.4) is nothing but, the area-preserving diffeomorphism algebra of a usual torus, SDiff(T^2), which is identified with SU(∞) [16]. This is a rather simple justification of the Matrix model M2-brane dynamics correspondence [18]. For the general rational θ, one can check that for $\theta = q/N$, (3.4) again holds but

m_i, n_i are defined mod qN and it represents a subgroup of SU(∞) [1, 16]. In the irrational θ case, however the situation is a little different, in this case (3.3) can not be mapped to any subgroup of SU(∞).

Indeed (3.3) can be thought as the generalization of SDiff(T^2) to the NCT, SDiff(T^2_θ). In the usual geometric language, SDiff(T^2) is the set of the diffeomorphisms keeping the Kahler structure of the torus unchanged, $\rho = iV$, V the volume of the torus. In a more general case we know that the Kahler structure can have a real part: $\rho = iV + b$, and as recently discussed [4, 7, 8, 11, 12, 19], at least in the zero volume limit, b which in the string theory language is the background B-field flux, identified with the deformation parameter of the torus, θ. One can show that under the transformations generated by (3.3) algebra ρ is invariant. This fact supports arguments of [7, 8] about the U-duality group of string and brane theory with a non-zero B-field. Furthermore, (3.3) for the irrational θ can be treated as the area-preserving diffeomorphism algebra of the NCT, SDiff(${T_\theta}^2$).

The gauge theory action. Now, we imply (3.1) in writing the action (2.18) in momentum space in terms of $L_{\vec{k}}$ and Fourier modes. Here we perform calculations with the non-supersymmetric action, its generalization to the supersymmetric case, obtained from (2.20), is like the usual gauge theories, thus we only quote the results.

Plugging (3.1) into the (2.17), we have

$$F_{\mu\nu} = \sum_{\vec{k}} -ik_{[\mu}A_{\nu]}(k)L_{\vec{k}} + 2i\sum_{\vec{m},\vec{n}} \sin(\pi\theta\vec{m}\times\vec{n})A_\mu(m)A_\nu(n)L_{\vec{m}+\vec{n}} \tag{3.5}$$

and hence the action of the U(1) NCYM, (2.18), in Fourier modes is:

$$\begin{aligned} S = \frac{1}{g^2_{YM}} & \sum_{\vec{k}}(-k_{[\mu}A_{\nu]}(k))(k_{[\mu}A_{\nu]}(-k)) + \\ & +2\sum_{\vec{m},\vec{n},\vec{k}} k_{[\mu}A_{\nu]}(-k)\sin(\pi\theta\vec{m}\times\vec{n})A_\mu(m)A_\nu(n)\delta(\vec{m}+\vec{n}+\vec{k}) + \\ & +4\sum_{\vec{m},\vec{n},\vec{k},\vec{l}} \sin(\pi\theta\vec{m}\times\vec{n})\sin(\pi\theta\vec{k}\times\vec{l})A_\mu(m)A_\nu(n)A_\mu(-k)A_\nu(-l)\delta(\vec{m}+\vec{n}+\vec{k}+\vec{l})\,. \end{aligned} \tag{3.6}$$

In writing the above action we have implied the reality condition of the fields, $A_\mu(-k) = A^*_\mu(k)$. The above action is of the form of the usual gauge theories in which group indices are identified with the momentum, the structure constants $f_{\vec{m},\vec{n},\vec{k}}$ are

$$f_{\vec{m},\vec{n},\vec{k}} = 2\sin(\pi\theta\vec{m}\times\vec{n})\delta(\vec{m}+\vec{n}+\vec{k})\,. \tag{3.7}$$

As we see the momentum conservation condition is automatically taken into account by the above structure constants. The important feature of this gauge theory is *gauge invariance*, (2.19), which enables us to handle the problem of renormalizability without caring about the non-locality of the theory.

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Noting the (3.7) and the action (3.6), we see that in this gauge theory our momentum dependent coupling is $\sin(\pi\theta\vec{m}\times\vec{n})$, which is smaller than one, hence we hope that the usual perturbative field theory methods, despite the momentum dependence of the couplings, work here.

Another point to mention here is the broken Lorentz invariance of (2+1) dimensional space. As it is seen from the action the couplings are a function of the *spatial* momentum only, which explicitly breaks the Lorentz invariance. The gauge fixing condition removes the subtleties related to the lack of Lorentz invariance. We will show later that, this Lorentz non-invariance due to the special form of the interactions will not destroy our perturbative calculations and is consistent with the gauge invariance at least, at one loop. Moreover we can find a "Ward identity" for this gauge theory.

Before going to the details of loop calculations, let us study the structure of the algebra defined by (3.3) and the related gauge theory, NCYM:

a) All the $L_{\vec{m}}$ with parallel $\vec{m}$ are commuting.

b) The only generator which commutes with all the others is $L_{\vec{0}}$, so rank of the group is one and the related Cartan sub-algebra is U(1).

c) In the usual non-Abelian gauge theories, the gauge particles are charged under the Cartan sub-algebra, in our case since the Cartan is U(1), the gauge particles are "*dipoles*" under the sub-algebra [19], and the corresponding dipole moments are proportional to their *momentum* but always *normal* to it and parallel dipoles are non-interacting.

d) Because the momentum is conserved in each vertex, here we have the *dipole moment conservation.*

e) From the string theory point of view [19], these dipoles are lowest modes of the strings attached to a D2-brane in a B-field background.

f) The high energy dipole-dipole scattering is suppressed by the Moyal bracket structure.

Perturbative tools. To do the calculations we need to read off the propagators and interaction vertices from the action (3.6). For further details we refer the reader to [20].

4. Loop calculations and renormalization

In this section, in order to discuss the renormalizability, β-function behaviour of the theory and its UV and IR limits, we explicitly calculate the one loop diagrams of the NCYM. For simplicity, we take the continuum limit and instead of summation we use integrals. The theory is supposed to have a well-defined continuum limit [13].

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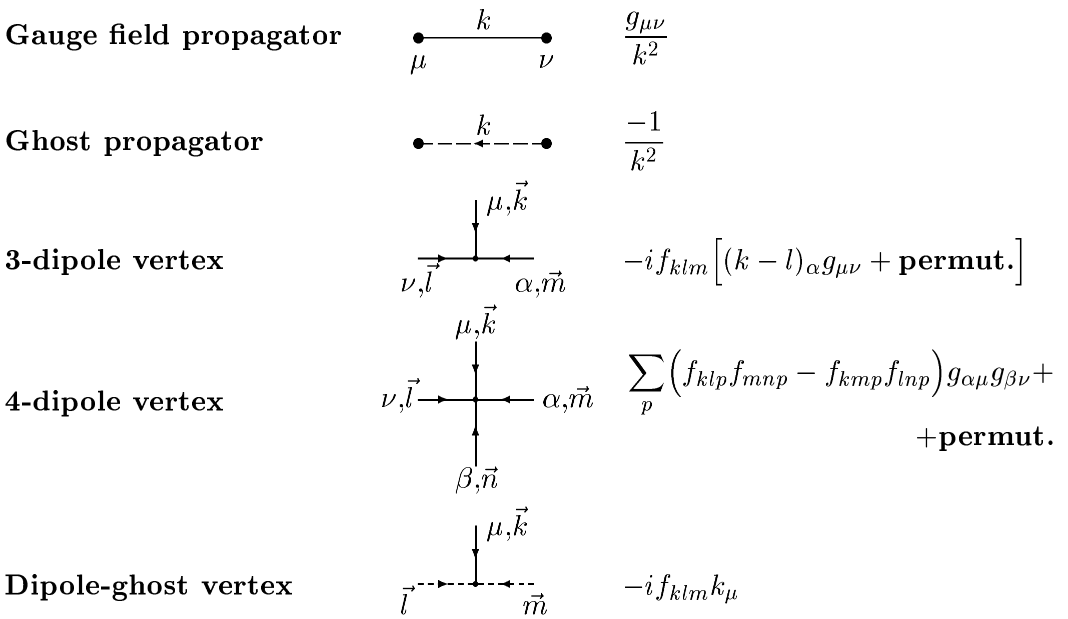

Figure 1: Basic Feynmann diagrams

Loop calculations

1. The dipoles self energy part:

   it gets contributions from four diagrams, dipole-loop, dipoles tadpole-loop, ghost-loop and the counter-term [20].

   1.1. Dipole-loop:

$$\Pi_{\mu\nu}(k) = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{1}{q^2(q+k)^2} A_{\mu\nu}\left(2\sin(\pi\theta\vec{k}\times\vec{q})\right)^2 , \tag{4.1}$$

with

$$A_{\mu\nu} = \left(q^2 + (q+k)^2 + 4k^2\right) g_{\mu\nu} + 6q_\mu q_\nu + 3(q_\mu k_\nu + q_\nu k_\mu) - 3k_\mu k_\nu\,. \tag{4.2}$$

It is worth noting that, in the case of NCYM theories, unlike the usual loop calculations, since the theory is not Lorentz invariant, we can not use the Wick rotation method. To perform the loop integrals, since there is no q_0 dependence in the vertex functions, first we integrate over the q_0 component by taking the residue of q_0 poles. The remaining integrals are of the form:

$$\Pi_{\mu\nu}(k) = \frac{4}{2}\int\frac{qdqd\phi}{(2\pi)^3}\left[\frac{A_{\mu\nu}|_{q_0=|q|}}{(k^2+2kq\cos\phi)} - \frac{A_{\mu\nu}|_{q_0=-|q|}}{(k^2-2kq\cos\phi)}\right]\sin(\pi\theta kq\sin\phi)^2, \tag{4.3}$$

Before performing the integration over ϕ, we note that

$$\sin^2(\pi k q\theta \sin\phi) = \frac{1}{2} - \frac{1}{2}\cos(2\pi\theta k q \sin\phi)\,.$$

The cosine part can be expanded in terms of Bessel functions [21]. It is a straightforward, but messy calculation to show that all the terms containing Bessel functions are finite, and the only divergent part comes from the 1/2 term of the sine squared. So, to find the divergent part of the integral it is enough to substitute the sine squared in (4.1) for 1/2. In other words, the divergent part of (4.1) has exactly the form of the usual SU(N) YM theory with $C_2(G) = 4 \times 1/2$. Furthermore, the divergent part of (4.1) is *Lorentz invariant* and can be calculated by Wick rotation.

1.2. Dipole tadpole- and Ghost- loops:

using the method explained earlier, i.e. integrating on q_0 component and expanding the sine squared in terms of Bessel functions, one can show that the divergent part of these diagrams, like the dipole-loop case gets contributions only from putting the sine squared equal to 1/2, which again restores the *Lorentz invariance* of the theory.

So altogether the dipole self energy part, is renormalized like a usual local gauge theory, with $C_2(G) = 2$. The other important result is that although the interactions introduced by NCYM theory are not Lorentz invariant, the Lorentz invariance *remains* in the propagators at one loop.

2. Ghost self energy part:

there are only two diagrams contributing here, the ghost-dipole loop and the counter-term. The integrals appearing in ghost-dipole loop, is similar to the ghost-loop contributions discussed above, hence our argument holds true and the divergent part is obtained from the usual SU(N) gauge theory with $C_2(G) = 2$.

The other important result we obtain here is a generalized "Ward identity",

$$k^\mu \Pi_{\mu\nu} = 0\,.$$

3. Dipole-ghost vertex:

there are three diagrams contributing here, the diagram with two-ghosts in the loop, the diagram with one ghost in the loop and the counter-term. Performing calculations by the method explained above, after a long but straight-forward calculus, we again find that the *divergent part* of the loops is described by usual gauge theory result with $C_2(G) = 2$.

For the three and four dipole vertices the same results are obtained. We see from explicit loop calculations that the divergent part of the NCYM theory at one loop, is governed by the divergent part of the loops of a usual gauge theory. Hence the divergent parts behave like a local field theory. However there are finite contributions at one loop level which are *non-local.* These terms also break the "Lorentz invariance". Appearance of such terms suggests the possibility of non-local divergent terms at higher loops [12]. But, according to our calculations [22], due to the Bessel functions, the contribution of the non-local terms at *two loop* are again suppressed.

We should remind here that, in all of the calculations we have assumed that the structure constant, $2\sin(\pi\theta\vec{m}\times\vec{n})$, is non-zero. So, although our result is θ independent, the above arguments are not valid for $\theta=0$ case.[2]

5. UV and IR behaviour of noncommutative gauge theories

By explicit loop calculations, it was shown in previous section that the NCYM theory is one loop renormalizable. Here we want to discuss the UV and IR limits in more detail. Since we consider the theory on a torus, however in the large volume limit, p, the momentum, is discrete and we need not to address the $p\to 0$ behaviour.

According to our results, although the theory is non-local and hence the usual renormalization group arguments are not valid, the one loop β-function of NCYM is negative, it is asymptotically free. Moreover in the large volume limit, the UV disentangles from the IR [13].

In the UV limit, besides the formal arguments of [13] our explicit calculations show that, we deal with a *non-local* theory. In UV, although the theory is not scale invariant, it admits a fixed point.

The supersymmetric case, NCSYM, is realized by the D2-brane in a B-field background [2, 11] which in the $\alpha'\to 0$, the low energy dynamics, like the usual D-brane arguments, decouples from the bulk. So the collective coordinates of the D2-brane, which are living in a commuting space, form a single vector multiplet. The 7 scalars correspond to 7 transverse direction of the brane. The dual of the vector multiplet, which is a scalar corresponds to the position of the brane in the eleventh dimension [23]. Hence the moduli space of the U(1) NCSYM theory is $R^7\times S^1$. The coupling of this (2+1) dimensional gauge theory is given by the circumference of the S^1 factor. Besides the transverse coordinates, since we are interested in the gauge theories on NCT, there are moduli coming from the compact space and noncommutative structure of the tours. In the string theory limit, these moduli are those the T-duality group acting on

$$\mathcal{M}_c=\frac{\mathrm{SL}(2)_N}{\mathrm{SO}(2)}\times\frac{\mathrm{SL}(2)_c}{\mathrm{SO}(2)}\,. \tag{5.1}$$

[2]In the rational θ case always one can find a basis on the torus under which the structure constant is zero.

The $\mathrm{SL}(2)_N$ acts on the deformation parameter of the torus and the other on the corresponding radii. Considering the contribution from the flat space we have

$$\mathcal{M} = \mathcal{M}_c \times R^7 \times S^1 \tag{5.2}$$

At strong coupling the S^1 factor combines with $\mathrm{SL}(2)_c$ and hence [8]

$$\mathcal{M} = \frac{\mathrm{SL}(2)_N}{\mathrm{SO}(2)} \times \frac{\mathrm{SL}(3)_c}{\mathrm{SO}(3)} \times R^7 \,. \tag{5.3}$$

6. Concluding remarks

In this paper, we have studied the SYM on the NCT more extensively. The NCSYM is described by the usual gauge theory action, with the commutators substituted for the *Moyal bracket*. As discussed in [2, 11], the NCSYM describes the low energy dynamics of D2-brane wrapped on T^2 in a B-field background, and in $\alpha' \to 0$ it decouples from the bulk. Studying the NCT, we argued that by means of *Morita equivalence* we can map the $\mathrm{U}(N)$ NCSYM with a magnetic flux M to a $\mathrm{U}(1)$ NCSYM theory, hence we only considered the $\mathrm{U}(1)$ case in our later arguments.

Using Fourier mode expansion, we wrote the Moyal bracket algebra in terms of Fourier basis, (3.3). The key idea, is the existence of commutative derivatives, could be defined on the NCT. Hence one expects by going to the momentum space, the NCYM translate to a commutative gauge theory. We argued that the gauge group is the generalization of the area-preserving diffeomorphisms of the torus to the NCT, $SDiff(T^2_\theta)$. As discussed, this gauge group do not change the Kahler structure of the NCT. Writing the NCYM action in the Fourier modes, we showed that it is like a usual gauge theory, with the structure constants given by (3.7). The group indices are identified with the momentum.

We showed that the Cartan sub-algebra in our case is $\mathrm{U}(1)$, and hence justified the string theoretic arguments of [19]. The other gauge particles are "dipoles" under this $\mathrm{U}(1)$ part. Having the action, we worked out the loop calculations, and found out that

i) Although the NCYM is not a local gauge theory in the usual sense and the interactions introduced in it are not Lorentz invariant, the divergent parts of the propagator loops are Lorentz invariant and the divergent part of the interaction vertices have the same structure of the classical ones. This enables us to make the statements we already know about the usual gauge theories, for the NCYM too. The divergent parts of the loops are given by the usual gauge theory results with the quadratic Casimir equal to 2. Moreover we can have generalized "Ward identities" here.

ii) The non-locality in the theory, is not removed in the loops, but at one loop, since the theory is renormalizable, it has the same non-local structure. In other words the structure constants of the theory are *not renormalized*, like the usual gauge theories. It seems that, the contribution of the non-local terms to the dipoles self energy are finite and also suppressed at high energies so that, they will not give further contributions at higher loops [22].

iii) Since the loop corrections are similar to the usual SU(N) gauge theories, the NCYM theories are *asymptotically* flat and in spite of the non-locality, the theory has got a UV fixed point.

iv) NCYM are among the few gauge theories which admit the maximally super extension. As argued here, these theories are *exact*. The moduli space structure of these theories, compared with the usual (2+1) $\mathcal{N}=8$ SYM, have an extra SL(2) factor acting on the noncommutative structure of the torus. Since this extra factor changes the deformation parameter θ, the structure constants of the theory are also changed, but its behaviour is not altered.

The possible singular structure of the moduli space, is an open question we will postpone it to a future work.

When I was typing the manuscript, [24] appeared on the net in which the same problem has been considered.

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